Eur. Phys. J. B **14**, 661–667 (2000)

THE EUROPEAN PHYSICAL JOURNAL B
EDP Sciences

Effects of random biquadratic couplings in a spin-1 spin-glass model

J.M. de Araújo[1,2], F.A. da Costa[2,a], and F.D. Nobre[2]

[1] Departamento de Ciências Naturais, Universidade Estadual do Rio Grande do Norte, 59610-210 Mossoró - RN, Brazil
[2] Departamento de Física Teórica e Experimental, Universidade Federal do Rio Grande do Norte, Campus Universitário, C.P. 1641, 59072-970 Natal - RN, Brazil

Received 18 May 1999 and Received in final form 20 October 1999

Abstract. A spin-1 model, appropriated to study the competition between bilinear ($J_{ij}S_iS_j$) and biquadratic ($K_{ij}S_i^2S_j^2$) random interactions, both of them with zero mean, is investigated. The interactions are infinite-ranged and the replica method is employed. Within the replica-symmetric assumption, the system presents two phases, namely, paramagnetic and spin-glass, separated by a continuous transition line. The stability analysis of the replica-symmetric solution yields, besides the usual instability associated with the spin-glass ordering, a new phase due to the random biquadratic couplings between the spins.

PACS. 05.70.-a Thermodynamics – 05.70.Fh Phase transitions: general studies – 64.60.-i General studies of phase transitions

1 Introduction

The study of disordered systems has grown very fast during the last years. Among these systems, spin glasses [1–3] have attracted much attention. One of its main characteristic is the existence of a very rugged free-energy landscape, with many minima separated by high barriers. It turns out that the equilibrium state of such system becomes hardly accessible in an experiment, as one may guess. The spin-glass mean-field theory is well-established [2,3], being highly nontrivial. However, the effects of fluctuations around the mean-field solution are very difficult to take into account in general cases, with most of the results been obtained for the Edwards-Anderson model [4].

Many other spin-glass models have been investigated within the mean-field level (for reviews, see Refs. [1–3]). Recently, much effort has been dedicated to understanding the phase behavior of spin-1 Ising glasses [5–9], as promising models to describe real systems which present multicritical phenomena. Other models which can be mapped onto spin-1 Ising glasses were also studied recently [10–17]. However, as far as we know, none of those works addresses to the competition between bilinear and biquadratic random interactions. In fact, a few years ago a spin-glass version of the Blume-Emery-Griffiths model [18] was introduced in order to describe disordered magnetic lattice gases [19–23], including both bilinear and biquadratic random couplings; however, due to the fact that a replica stability analysis was not performed, an important ingredient was missing, *i.e.*, broken ergodicity, usually associated with irreversibility effects. The purpose of the present work is to fill this gap by investigating the overall behavior of a system which presents the aforementioned random interactions in a simple spin-1 model. In order to determine the free-energy density and the corresponding equations of state, we will use the replica mean-field approach. Under the replica-symmetry assumption [24], the system exhibits only two phases separated by a continuous transition line. However, the stability analysis of the replica-symmetric solution performed within the approach proposed by de Almeida and Thouless [25] suggests the existence of three distinct phases. The paper is organized as follows. In Section 2 we describe the model and obtain the replica free energy. The replica-symmetric solution, as well as the corresponding phase diagram is investigated in Section 3. The stability analysis of the replica-symmetric solution is performed in Section 4. Our findings are summarized in Section 5, where we also present our conclusions.

2 The model and its free-energy density

In this paper we consider an infinite-range interaction spin-glass model described by the Hamiltonian

$$H = -\sum_{(i,j)} J_{ij}S_iS_j - \sum_{(i,j)} K_{ij}S_i^2S_j^2, \qquad (1)$$

[a] *Present address*: Instituto de Física, Universidade Federal Fluminense, Av. Litorânea s/n – Boa Viagem, 24210-340 Niteroi - RJ, Brazil.
e-mail: `fcosta@dfte.ufrn.br`

where each spin S_i $(i = 1, 2, \cdots, N)$ can take the values $-1, 0$ and 1 and the summations are over all distinct pairs (i, j). Both couplings are quenched, independent random variables, following probability distributions

$$P(X_{ij}) = \left(\frac{N}{2\pi X^2}\right)^{1/2} \exp\left(-\frac{N X_{ij}^2}{2X^2}\right), \tag{2}$$

where X stands for either J or K. There are two obvious limiting cases of this model. First, in the absence of biquadratic interactions ($K_{ij} = 0$ for every pair (i, j)), we have a conventional spin-1 spin-glass model. The properties of this model are quite analogous to those of the Sherrington-Kirkpatrick model [5–9,26–28]; the system presents a continuous transition from a paramagnetic to a low-temperature spin-glass phase where the ergodicity is also broken [27]. On the other hand, if $J_{ij} = 0$ for every pair (i, j), the system becomes equivalent to the discrete quadrupolar-glass model investigated in reference [10]. In this case, there is no sharp transition to a low-temperature phase; however, a stability analysis shows that in fact there is a phase transition to a low-temperature nonergodic region [11]. We will be most interested in the case where both J_{ij} and K_{ij} are distinct from zero, in order to appreciate the effects of their competition on the phase diagram.

The free-energy density for this system is given by

$$\beta f = -\lim_{N\to\infty} \frac{\overline{\ln Z}}{N}, \tag{3}$$

where the bar denotes an average over the disorder. Such an average is performed by the so-called replica method, through the identity

$$\ln Z = \lim_{n\to 0} \frac{Z^n - 1}{n}, \tag{4}$$

which avoids the difficulty of averaging the logarithm. Using standard procedures [2,3], we obtain

$$\beta f = \lim_{n\to 0} \frac{1}{n} \min g_n(q_{\alpha\beta}, Q_{\alpha\beta}, p_\alpha), \tag{5}$$

where

$$g_n(q_{\alpha\beta}, Q_{\alpha\beta}, p_\alpha) = \frac{1}{4}\sum_{\alpha\neq\beta}\left(\beta^2 J^2 q_{\alpha\beta}^2 + \beta^2 K^2 Q_{\alpha\beta}^2\right) + \frac{\beta^2}{4}(J^2+K^2)\sum_\alpha p_\alpha^2 - \ln \mathrm{Tr}\exp(\mathcal{H}_{\mathrm{eff}}) \tag{6}$$

and

$$\mathcal{H}_{\mathrm{eff}} = \frac{\beta^2 J^2}{2}\sum_{\alpha\neq\beta} q_{\alpha\beta} S^\alpha S^\beta + \frac{\beta^2 K^2}{2}\sum_{\alpha\neq\beta} Q_{\alpha\beta}\left(S^\alpha\right)^2\left(S^\beta\right)^2 + \frac{\beta^2}{2}\left(J^2+K^2\right)\sum_\alpha p_\alpha^2\left(S^\alpha\right)^2, \tag{7}$$

with the indexes α and β running from 1 to n. Stationarity of g_n with respect to $q_{\alpha\beta}$, $Q_{\alpha\beta}$ and p_α gives the equations of state,

$$\begin{aligned} p_\alpha &= \left\langle \left(S^\alpha\right)^2 \right\rangle_n, \\ q_{\alpha\beta} &= \left\langle S^\alpha S^\beta \right\rangle_n, \\ Q_{\alpha\beta} &= \left\langle (S^\alpha S^\beta)^2 \right\rangle_n, \end{aligned} \tag{8}$$

where $\langle\ \rangle_n$ denotes an average with respect to the "effective Hamiltonian" in equation (7). Whereas the order parameters $q_{\alpha\beta}$ and $Q_{\alpha\beta}$ are already expected, the free energy depends also on p_α, a disorder induced order parameter which measures the fraction of spins in the states $S^\alpha = \pm 1$, for each replica α.

In the following two sections we consider the replica-symmetric solution and its corresponding stability analysis.

3 Replica-symmetric solution

The simplest solution of the saddle-point equations is the replica symmetric ansatz, which consists in assuming

$$\begin{aligned} p_\alpha &= p, \qquad \forall\alpha \\ q_{\alpha\beta} &= q, \qquad \forall(\alpha\beta) \\ Q_{\alpha\beta} &= Q, \qquad \forall(\alpha\beta). \end{aligned} \tag{9}$$

Inserting this ansatz into equations (5–7) and performing some simple Gaussian transformations, the free-energy density becomes

$$f = \frac{\beta J^2}{4}\left(p^2 - q^2\right) + \frac{\beta K^2}{4}\left(p^2 - Q^2\right) - \langle\langle \ln z(x,y)\rangle\rangle_{xy} \tag{10}$$

where

$$z(x,y) = 1 + 2\exp(\varDelta)\cosh\left(\beta J\sqrt{q}x\right), \tag{11}$$

$$\varDelta = \frac{\beta^2 J^2}{2}(p-q) + \frac{\beta^2 K^2}{2}(p-Q) + \beta K\sqrt{Q}y, \tag{12}$$

and

$$\langle\langle h(x,y)\rangle\rangle_{xy} = \int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{\sqrt{2\pi}} \times \int_{-\infty}^{+\infty}\frac{\mathrm{d}y}{\sqrt{2\pi}}\exp\left(-\frac{x^2+y^2}{2}\right)h(x,y). \tag{13}$$

For the equations of state one gets,

$$p = \langle\langle \varphi_2(x,y)\rangle\rangle_{xy}, \tag{14}$$

$$q = \left\langle\left\langle \varphi_1^2(x,y)\right\rangle\right\rangle_{xy}, \tag{15}$$

$$Q = \left\langle\left\langle \varphi_2^2(x,y)\right\rangle\right\rangle_{xy}, \tag{16}$$

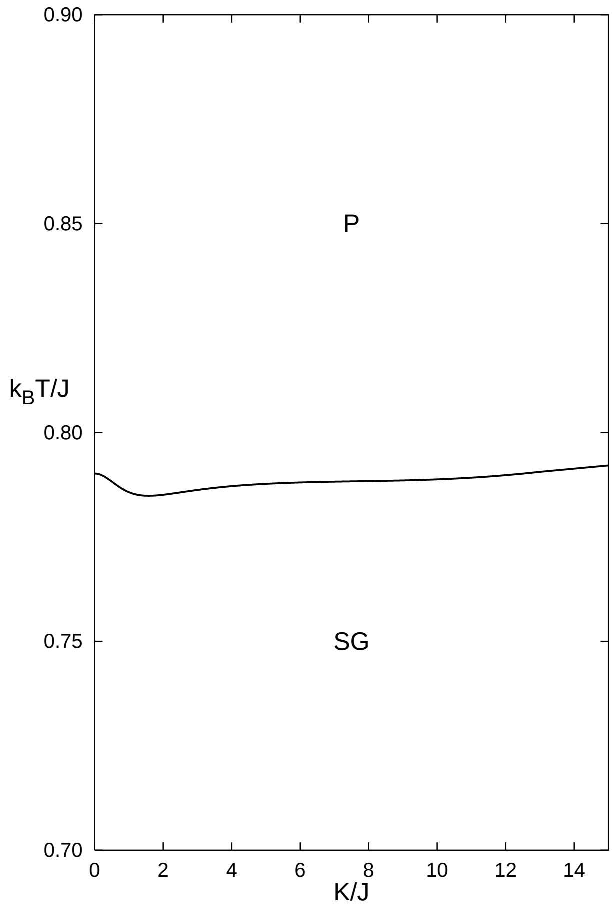

Fig. 1. Phase diagram obtained within the replica-symmetric solution, presenting a continuous transition from a high-temperature paramagnetic phase (P) to a low-temperature spin-glass phase (SG).

with

$$\varphi_1(x,y) = \frac{2\mathrm{e}^{\Delta}}{z(x,y)} \sinh(\beta J \sqrt{q} x), \tag{17}$$

$$\varphi_2(x,y) = \frac{2\mathrm{e}^{\Delta}}{z(x,y)} \cosh(\beta J \sqrt{q} x). \tag{18}$$

As mentioned before, the $K = 0$ case is equivalent to $D = 0$ in the model studied by Ghatak and Sherrington [26]. In this particular case it is well-known that we have a continuous transition from a paramagnetic to a spin-glass phase at $k_{\mathrm{B}}T/J = 0.7901\cdots$. In general, the above equations present a trivial solution $q = 0$ but with $p \neq 0$, $Q \neq 0$ for any temperature. In the low-temperature regime we also find another phase, with all order parameters distinct from zero. The phase boundary separating these two phases can be easily obtained from an expansion in powers of q, in either the free-energy density, or the equation of state for q. In either way, we find a critical frontier given by

$$Q = (k_{\mathrm{B}}T/J)^2. \tag{19}$$

From the same expansions, we also ruled out the possibility of first-order transitions and tricritical behavior. The condition given by the above equation involves both order parameters Q and p, which should satisfy equations (14, 16) with q set to zero. We are thus left with a set of three coupled nonlinear equations, which, except for some particular limits, has no analytical solution. We performed a detailed numerical study of these equations in order to check for the possibility of other types of orderings, but we found none, besides those already described. As a result of our analysis, we found the critical frontier shown in Figure 1. We performed an expansion for $K \gg J$, and verified that asymptotically, such a critical frontier approaches the limit $k_{\mathrm{B}}T/J = 0.7876\cdots$, with $p \cong Q = 0.6204\cdots$. It is important to mention that within the present analysis, the high-temperature phase should be identified as an extension of the paramagnetic one already present when $K = 0$. This can be justified by the following argument: the free-energy density, as well as the order parameters p and Q, may be expanded as power series of K, for small values of K. Therefore, no anomalous behavior on the thermodynamical functions can be seen as we let $K \to 0$. Similarly, the low-temperature phase should be identified with the spin-glass phase occurring at $K = 0$. Thus, as far as the replica symmetric solution is concerned, the biquadratic random coupling does not bring any new physics to this system.

In the following section we will consider the stability of the above-mentioned solutions against replica fluctuations, and it will be shown that this study leads to an important modification on the paramagnetic side.

4 Stability of the replica-symmetric solutions

Since the work of de Almeida and Thouless [25], it is generally believed that replica-symmetric solutions are unstable under small fluctuations on the whole replica space. In our case, these fluctuations are governed by the Hessian matrix,

$$\mathbf{G} = \begin{pmatrix} \dfrac{\partial^2 g_n}{\partial p_\alpha \partial p_\beta} & \dfrac{\partial^2 g_n}{\partial p_\alpha \partial q_{\nu\gamma}} & \dfrac{\partial^2 g_n}{\partial p_\alpha \partial Q_{\nu\gamma}} \\ \dfrac{\partial^2 g_n}{\partial q_{\nu\gamma} \partial p_\alpha} & \dfrac{\partial^2 g_n}{\partial q_{\alpha\beta} \partial q_{\nu\gamma}} & \dfrac{\partial^2 g_n}{\partial q_{\alpha\beta} \partial Q_{\nu\gamma}} \\ \dfrac{\partial^2 g_n}{\partial Q_{\nu\gamma} \partial p_\alpha} & \dfrac{\partial^2 g_n}{\partial Q_{\nu\gamma} \partial q_{\alpha\beta}} & \dfrac{\partial^2 g_n}{\partial Q_{\alpha\beta} \partial Q_{\nu\gamma}} \end{pmatrix} \tag{20}$$

where g_n is given by equation (6). Stability requires that all eigenvalues of this matrix, evaluated within the replica-symmetric solution, should be positive (see Appendix A for the computation of such eigenvalues). In the limit $n \to 0$ we get three longitudinal eigenvalues, as the roots

$$\begin{vmatrix} \mathcal{A}-\mathcal{B}-\lambda^{(\mathrm{L})} & \mathcal{D}-\mathcal{C} & \mathcal{F}-\mathcal{E} \\ 2(\mathcal{C}-\mathcal{D}) & \mathcal{G}-4\mathcal{H}+3\mathcal{I}-\lambda^{(\mathrm{L})} & \mathcal{J}-4\mathcal{K}+3\mathcal{L} \\ 2(\mathcal{F}-\mathcal{E}) & \mathcal{J}-4\mathcal{K}+3\mathcal{L} & \mathcal{M}-4\mathcal{N}+3\mathcal{O}-\lambda^{(\mathrm{L})} \end{vmatrix} = 0\ , \tag{21}$$

$$\lambda^{(\mathrm{T})} = \frac{\mathcal{G}-2\mathcal{H}+\mathcal{I}+\mathcal{M}-2\mathcal{N}+\mathcal{O} \pm \sqrt{[\mathcal{G}-2\mathcal{H}+\mathcal{I}-\mathcal{M}+2\mathcal{N}-\mathcal{O}]^2-4[\mathcal{J}-2\mathcal{K}+\mathcal{L}]^2}}{2} \tag{22}$$

$$\lambda^{(\mathrm{L})}_{2,3} = \frac{\mathcal{A}-\mathcal{B}+\mathcal{M}-4\mathcal{N}+3\mathcal{O} \pm \sqrt{[\mathcal{A}-\mathcal{B}-\mathcal{M}+4\mathcal{N}-3\mathcal{O}]^2-8(\mathcal{E}-\mathcal{F})^2}}{2}\ . \tag{24}$$

of the secular equation

see equation (21) above

and two transverse ones, given by

see equation (22) above

where the quantities $\mathcal{A},\ldots,\mathcal{O}$ are defined in Appendix A. In the paramagnetic phase, where $q=0$, one of the longitudinal eigenvalues becomes

$$\lambda_1^{(\mathrm{L})} = \left(\frac{J}{k_{\mathrm{B}}T}\right)^2 - \left(\frac{J}{k_{\mathrm{B}}T}\right)^4 Q\ , \tag{23}$$

whereas the other two are given by

see equation (24) above.

Let us first consider the behavior of the above eigenvalues throughout the paramagnetic phase. Using both analytical and numerical calculations, we find that all three longitudinal eigenvalues are positive, with $\lambda_1^{(\mathrm{L})}$ vanishing along the paramagnetic to spin-glass transition line. However, considering the transverse eigenvalues of equation (22), we notice that one of them, denoted by $\lambda_1^{(\mathrm{T})}$ (corresponding to the plus sign before the square root), becomes identical to $\lambda_1^{(\mathrm{L})}$ everywhere in the paramagnetic phase, including the critical frontier paramagnetic/spin-glass, where it also vanishes. Besides, in the paramagnetic region the second transverse eigenvalue is given by

$$\lambda_2^{(\mathrm{T})} = \mathcal{M}-2\mathcal{N}+\mathcal{O}\ . \tag{25}$$

Our numerical analysis shows that as the temperature decreases, and for high enough values of K, $\lambda_2^{(\mathrm{T})}$ becomes negative, throughout the paramagnetic phase. This suggests an onset of irreversibility in the paramagnetic phase, associated with an ergodicity breaking, as we cross the line given by $\lambda_2^{(\mathrm{T})}=0$. This effect is brought about by fluctuations on the order parameter $Q_{\alpha\beta}$, which in turn was generated by the random biquadratic couplings. We identify this region as a new phase, which we will call biquadratic spin-glass phase, with replica-symmetry breaking associated to the parameter $Q_{\alpha\beta}$; this region should, then, be properly described by the ansatz of Parisi [29]. We have also found that the boundary paramagnetic/biquadratic spin-glass (where $\lambda_2^{(\mathrm{T})}=0$) is a straight line with slope ≈ 0.077; such a numerical result is in full agreement with the one found in reference [11] ($k_{\mathrm{B}}T/J' \approx 1.38$), if one considers the proper changes of spin variables and summations in the Hamiltonian of reference [11] (which leads to $K=18J'$).

We have also investigated numerically the behavior of all five eigenvalues in the spin-glass phase. The transverse eigenvalue $\lambda_1^{(\mathrm{T})}$ is negative through the whole spin-glass phase; this means that irreversibility is also present in this phase and so, a solution with replica-symmetry breaking should be employed.

The phase diagram resulting from this analysis is shown in Figure 2. The paramagnetic to spin-glass as well as the paramagnetic to biquadratic spin-glass phase boundaries should remain valid under a Parisi-like treatment. However, it is possible that the biquadratic spin-glass to spin-glass frontier changes under replica-symmetry breaking in both matrices $\mathbf{Q}$ and $\mathbf{q}$. Thus, the corresponding boundary shown in Figure 2, which was obtained within the replica-symmetric solution, should be seen as a rather schematic one, although there is no physical reason to expect a substantial qualitative change. The correct treatment based on Parisi's ansatz is very difficult in this case, since it involves nonlinear integro-differential equations at finite temperatures, which are hard to solve numerically. Such an analysis is beyond the purpose of this paper.

5 Conclusions

We have studied a solvable spin-1 model, including both bilinear and biquadratic random exchanges, with zero means and variances J and K, respectively. The model was solved through the replica formalism. Three types of order parameters were introduced to describe the system in the replica space: a density (p_α), which measures the fraction of spins in the states $S^\alpha = \pm 1$ and two

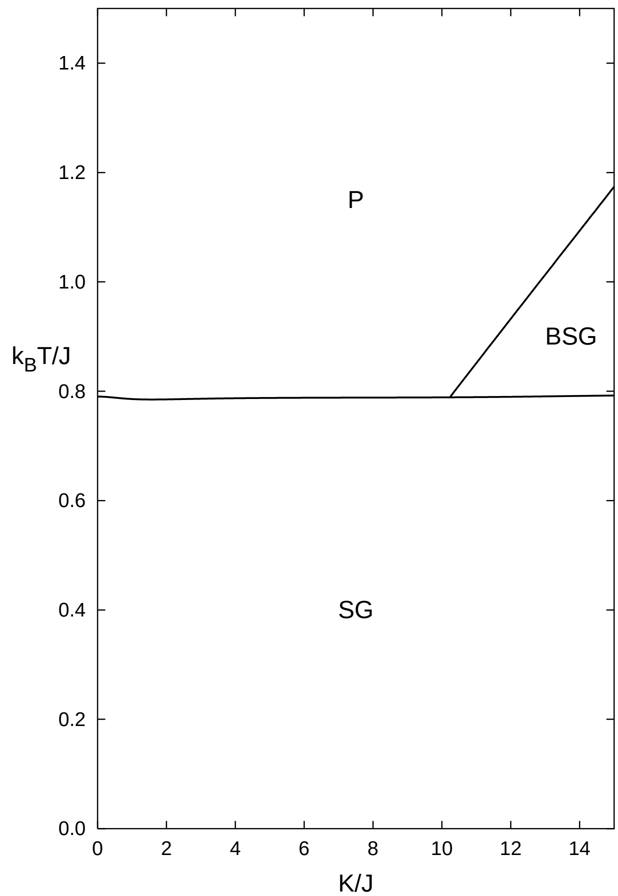

Fig. 2. Phase diagram resulting from the stability analysis of the replica-symmetric solution. This solution is stable throughout the paramagnetic (P) phase only. The biquadratic spin-glass phase (BSG), which occurs for large values of K, is characterized by an instability of the replica-symmetric paramagnetic solution. In the whole spin-glass phase (SG) the replica-symmetric solution is unstable.

spin-glass-like matrices, represented by the bilinear and biquadratic matrix elements $q_{\alpha\beta}$ and $Q_{\alpha\beta}$, respectively.

The replica-symmetric solution leads to a continuous transition from a high-temperature paramagnetic phase to a low-temperature spin-glass phase, signaled by the onset of the spin-glass order parameter q. The corresponding critical frontier is almost temperature-independent, especially for large values of the variance K. We have also analysed the eigenvalues of the stability matrix associated with fluctuations around the replica-symmetric solutions. We verified numerically that one of the replicon eigenvalues is always negative throughout the whole spin-glass phase, implying an instability of the replica-symmetric solution. We have also noticed that the paramagnetic phase presents a similar instability (associated with the matrix elements $Q_{\alpha\beta}$), for sufficiently large values of the variance K, giving rise to a new phase which we have called biquadratic spin-glass phase. Such instabilities may be related to the onset of irreversibility effects, *i.e.*, the response functions could depend on the history of the system (*e.g.*, field-cooling and zero-field-cooling measurements may lead to different results) [2,3]. On the basis of our findings, many of them from numerical analysis, we conclude that for $K \neq 0$ the system presents at least three distinct phases, in which two of them should be properly described through a replica-symmetry breaking procedure. The frontier separating the biquadratic spin-glass and spin-glass phases requires further investigation. It is not clear if a full Parisi solution would change substantially its location. We hope to address to this point in a future work.

One of us (FDN) thanks CNPq and Pronex/MCT (Brazil) for partial financial support. FAC would like to thank FAPESP for partial financial support during his visit to the Instituto de Física of the Universidade de São Paulo under the grant #97/14223-7. Finally, all of us would like to thank the hospitality of the Centro Brasileiro de Pesquisas Físicas.

Appendix A: Stability analysis of the replica-symmetric solution

The elements of the Hessian matrix $\mathbf{G}$, defined in equation (20), are given by

$$\frac{\partial^2 g_n}{\partial p_\alpha \partial p_\beta} = \frac{1}{2}(\beta\kappa)^2\delta_{\alpha\beta} - \frac{1}{4}(\beta\kappa)^4\left[\left\langle (S^\alpha S^\beta)^2\right\rangle_n - \left\langle (S^\alpha)^2\right\rangle_n \left\langle (S^\beta)^2\right\rangle_n\right], \tag{A.1}$$

$$\frac{\partial^2 g_n}{\partial p_\alpha \partial q_{\beta\gamma}} = -\frac{1}{2}(\beta\kappa)^2(\beta J)^2\left[\left\langle (S^\alpha)^2 S^\beta S^\gamma\right\rangle_n - \left\langle (S^\alpha)^2\right\rangle_n \left\langle S^\beta S^\gamma\right\rangle_n\right], \tag{A.2}$$

$$\frac{\partial^2 g_n}{\partial p_\alpha \partial Q_{\beta\gamma}} = -\frac{1}{2}(\beta\kappa)^2(\beta K)^2\left[\left\langle (S^\alpha S^\beta S^\gamma)^2\right\rangle_n - \left\langle (S^\alpha)^2\right\rangle_n \left\langle (S^\beta S^\gamma)^2\right\rangle_n\right], \tag{A.3}$$

$$\frac{\partial^2 g_n}{\partial q_{\alpha\beta} \partial q_{\gamma\delta}} = (\beta J)^2\delta_{\alpha\beta} - (\beta J)^4\left(\left\langle S^\alpha S^\beta S^\gamma S^\delta\right\rangle_n - \left\langle S^\alpha S^\beta\right\rangle_n \left\langle S^\gamma S^\delta\right\rangle_n\right), \tag{A.4}$$

$$\frac{\partial^2 g_n}{\partial q_{\alpha\beta} \partial Q_{\gamma\delta}} = -(\beta J)^2(\beta K)^2\left[\left\langle S^\alpha S^\beta (S^\gamma S^\delta)^2\right\rangle_n - \left\langle S^\alpha S^\beta\right\rangle_n \left\langle (S^\gamma S^\delta)^2\right\rangle_n\right], \tag{A.5}$$

$$\frac{\partial^2 g_n}{\partial Q_{\alpha\beta} \partial Q_{\gamma\delta}} = (\beta K)^2\delta_{\alpha\beta} - (\beta K)^4\left[\left\langle (S^\alpha S^\beta S^\gamma S^\delta)^2\right\rangle_n - \left\langle (S^\alpha S^\beta)^2\right\rangle_n \left\langle (S^\gamma S^\delta)^2\right\rangle_n\right], \tag{A.6}$$

where

$$\kappa^2 = J^2 + K^2. \tag{A.7}$$

For the replica-symmetric solution the eigenvectors of $\mathbf{G}$ have the form

$$\mathbf{u} = \begin{pmatrix} \epsilon_\alpha \\ \eta_{\alpha\beta} \\ \xi_{\alpha\beta} \end{pmatrix}, \tag{A.8}$$

where

$$\epsilon_\alpha = p_\alpha - p, \quad \eta_{\alpha\beta} = q_{\alpha\beta} - q, \quad \xi_{\alpha\beta} = Q_{\alpha\beta} - Q, \quad (A.9)$$

represent Gaussian fluctuations. Following de Almeida and Thouless [25], we start with the eigenvector totally symmetric under replica-index permutations

$$\epsilon_\alpha = a, \quad \eta_{\alpha\beta} = b, \quad \xi_{\alpha\beta} = c, \quad \text{for} \quad \alpha, \beta = 1 \dots n, \quad (A.10)$$

which correspond to the longitudinal eigenvectors, according to the conventional classification [30]. For a finite value of n, the corresponding eigenvalues follow from

$$\begin{aligned}
\lambda^{(\mathrm{L})} a &= \mathcal{A}a + (n-1)\mathcal{B}a + (n-1)\mathcal{C}b \\
&+ \frac{1}{2}(n-2)(n-1)\mathcal{D}b + (n-1)\mathcal{E}c \\
&+ \frac{1}{2}(n-2)(n-1)\mathcal{F}c, \qquad (A.11)\\
\lambda^{(\mathrm{L})} b &= 2\mathcal{C}a + (n-2)\mathcal{D}a + \mathcal{G}b + 2(n-2)\mathcal{H}b \\
&+ \frac{(n-2)(n-3)}{2}\mathcal{I}b + \mathcal{J}c + 2(n-2)\mathcal{K}c \\
&+ \frac{(n-2)(n-3)}{2}\mathcal{L}c, \qquad (A.12)\\
\lambda^{(\mathrm{L})} c &= 2\mathcal{E}a + (n-2)\mathcal{F}a + \mathcal{J}b + 2(n-2)\mathcal{K}b \\
&+ \frac{(n-2)(n-3)}{2}\mathcal{L}b + \mathcal{M}c + 2(n-2)\mathcal{N}c \\
&+ \frac{(n-2)(n-3)}{2}\mathcal{O}c, \qquad (A.13)
\end{aligned}$$

where

$$\mathcal{A} = \left.\frac{\partial^2 g_n}{\partial p_\alpha \partial p_\alpha}\right|_{\mathrm{RS}} = \frac{(\beta\kappa)^2}{2}\left[1 - \frac{(\beta\kappa)^2}{2}(1-p)p\right], \quad (A.14)$$

$$\mathcal{B} = \left.\frac{\partial^2 g_n}{\partial p_\alpha \partial p_\beta}\right|_{\mathrm{RS}} = \frac{(\beta\kappa)^4}{4}\left(p^2 - Q\right), \quad (A.15)$$

$$\mathcal{C} = \left.\frac{\partial^2 g_n}{\partial p_\alpha \partial q_{\alpha\beta}}\right|_{\mathrm{RS}} = \frac{(\beta J)^2(\beta\kappa)^2}{2}(p-1)q, \quad (A.16)$$

$$\mathcal{D} = \left.\frac{\partial^2 g_n}{\partial p_\alpha \partial q_{\beta\gamma}}\right|_{\mathrm{RS}} = \frac{(\beta J)^2(\beta\kappa)^2}{2}(pq - w), \quad (A.17)$$

$$\mathcal{E} = \left.\frac{\partial^2 g_n}{\partial p_\alpha \partial Q_{\alpha\beta}}\right|_{\mathrm{RS}} = \frac{(\beta\kappa)^2(\beta K)^2}{2}(p-1)Q, \quad (A.18)$$

$$\mathcal{F} = \left.\frac{\partial^2 g_n}{\partial p_\alpha \partial Q_{\beta\gamma}}\right|_{\mathrm{RS}} = \frac{(\beta\kappa)^2(\beta K)^2}{2}(pQ - W), \quad (A.19)$$

$$\mathcal{G} = \left.\frac{\partial^2 g_n}{\partial q_{\alpha\beta} \partial q_{\alpha\beta}}\right|_{\mathrm{RS}} = (\beta J)^2\left[1 + (\beta J)^2(q^2 - Q)\right], \quad (A.20)$$

$$\mathcal{H} = \left.\frac{\partial^2 g_n}{\partial q_{\alpha\beta} \partial q_{\alpha\gamma}}\right|_{\mathrm{RS}} = (\beta J)^4(q^2 - w), \quad (A.21)$$

$$\mathcal{I} = \left.\frac{\partial^2 g_n}{\partial q_{\alpha\beta} \partial q_{\gamma\delta}}\right|_{\mathrm{RS}} = (\beta J)^4(q^2 - s), \quad (A.22)$$

$$\mathcal{J} = \left.\frac{\partial^2 g_n}{\partial q_{\alpha\beta} \partial Q_{\alpha\beta}}\right|_{\mathrm{RS}} = (\beta J)^2(\beta K)^2(Q-1)q, \quad (A.23)$$

$$\mathcal{K} = \left.\frac{\partial^2 g_n}{\partial q_{\alpha\beta} \partial Q_{\alpha\gamma}}\right|_{\mathrm{RS}} = (\beta J)^2(\beta K)^2(Qq - w), \quad (A.24)$$

$$\mathcal{L} = \left.\frac{\partial^2 g_n}{\partial q_{\alpha\beta} \partial Q_{\gamma\delta}}\right|_{\mathrm{RS}} = (\beta J)^2(\beta K)^2(Qq - v), \quad (A.25)$$

$$\mathcal{M} = \left.\frac{\partial^2 g_n}{\partial Q_{\alpha\beta} \partial Q_{\alpha\beta}}\right|_{\mathrm{RS}} = (\beta K)^2\left[1 + (\beta K)^2(Q-1)Q\right], \quad (A.26)$$

$$\mathcal{N} = \left.\frac{\partial^2 g_n}{\partial Q_{\alpha\beta} \partial Q_{\alpha\gamma}}\right|_{\mathrm{RS}} = (\beta K)^4\left(Q^2 - W\right), \quad (A.27)$$

$$\mathcal{O} = \left.\frac{\partial^2 g_n}{\partial Q_{\alpha\beta} \partial Q_{\gamma\delta}}\right|_{\mathrm{RS}} = (\beta K)^4\left(Q^2 - S\right). \quad (A.28)$$

In the above equations, different replica labels are distinct from one another, whereas $..|_{\mathrm{RS}}$ means elements of the Hessian matrix evaluated within the replica-symmetric solution; the parameters p, q and Q are given by equations (14–16) and

$$S = \langle\langle \varphi_2^4 \rangle\rangle_{xy}, \quad (A.29)$$

$$s = \langle\langle \varphi_1^4 \rangle\rangle_{xy}, \quad (A.30)$$

$$v = \langle\langle \varphi_1^2 \varphi_2^2 \rangle\rangle_{xy}, \quad (A.31)$$

$$W = \langle\langle \varphi_2^3 \rangle\rangle_{xy}, \quad (A.32)$$

$$w = \langle\langle \varphi_1^2 \varphi_2 \rangle\rangle_{xy}. \quad (A.33)$$

The next step is to find the anomalous eigenvalues. These correspond to breaking the symmetry of the vector $\mathbf{u}$ with respect to one specific replica-index, denoted herein by θ,

$$\begin{cases} \epsilon^\alpha = a_1 & \text{for} \quad \alpha = \theta \\ \epsilon^\alpha = a_2 & \text{for} \quad \alpha \neq \theta \end{cases} \quad (A.34)$$

$$\begin{cases} \eta^{\alpha\beta} = b_1 & \text{for} \quad \alpha \text{ or } \beta = \theta \\ \eta^{\alpha\beta} = b_2 & \text{for} \quad \alpha, \beta \neq \theta \end{cases} \quad (A.35)$$

$$\begin{cases} \xi^{\alpha\beta} = c_1 & \text{for} \quad \alpha \text{ or } \beta = \theta \\ \xi^{\alpha\beta} = c_2 & \text{for} \quad \alpha, \beta \neq \theta. \end{cases} \quad (A.36)$$

Orthogonality with the replica-symmetric eigenvector implies

$$a_2 = -\frac{a_1}{n-1}, \quad b_2 = -\frac{b_1}{n-2}, \quad c_2 = -\frac{c_1}{n-2} \cdot \qquad \text{(A.37)}$$

Thus, the anomalous eigenvalues follow from

$$\lambda^{(\mathrm{A})} a_1 = (\mathcal{A} - \mathcal{B}) a_1 + (n-1)(\mathcal{C} - \mathcal{D}) b_1 + (n-1)(\mathcal{E} - \mathcal{F}) c_1, \qquad \text{(A.38)}$$

$$\lambda^{(\mathrm{A})} b_1 = \frac{n-2}{n-1}(\mathcal{C} - \mathcal{D}) a_1 + [\mathcal{G} + n\mathcal{H} - (n-3)\mathcal{I}] b_1 + [\mathcal{J} + n\mathcal{K} - (n-3)\mathcal{L}] c_1 = 0, \qquad \text{(A.39)}$$

$$\lambda^{(\mathrm{A})} c_1 = \frac{n-2}{n-1}(\mathcal{E} - \mathcal{F}) a_1 + [\mathcal{J} + n\mathcal{K} - (n-3)\mathcal{L}] b_1 + [\mathcal{M} + n\mathcal{N} - (n-3)\mathcal{O}] c_1 = 0. \qquad \text{(A.40)}$$

In the limit $n \to 0$ the longitudinal and anomalous eigenvalues coincide and may be obtained from equation (21).

Finally, we must find the transverse eigenvalues. In this case, two replica indices are fixed and the symmetry between replicas is broken with respect to such indices (herein denoted by θ and ν). One has,

$$\begin{cases} \epsilon^{\alpha} = a_3 & \text{for} \quad \alpha = \theta \ \text{ or } \ \nu \\ \epsilon^{\alpha} = a_4 & \text{for} \quad \alpha \neq \theta, \nu \end{cases} \qquad \text{(A.41)}$$

$$\begin{cases} \eta^{\alpha\beta} = b_3 & \text{for} \quad \alpha \ \text{ or } \ \beta = \theta \ \text{ or } \ \nu \\ \eta^{\theta\alpha} = \eta^{\nu\alpha} = b_4 & \text{for} \quad \alpha \neq \theta, \nu \\ \eta^{\alpha\beta} = b_5 & \text{for} \quad \alpha, \beta \neq \theta, \nu \end{cases} \qquad \text{(A.42)}$$

$$\begin{cases} \xi^{\alpha\beta} = c_3 & \text{for} \quad \alpha \ \text{ or} \beta = \theta \ \text{ or } \ \nu \\ \xi^{\theta\alpha} = \xi^{\nu\alpha} = c_4 & \text{for} \quad \alpha \neq \theta, \nu \\ \xi^{\alpha\beta} = c_5 & \text{for} \quad \alpha, \beta \neq \theta, \nu. \end{cases} \qquad \text{(A.43)}$$

These eigenvectors must be orthogonal to both longitudinal and anomalous ones. Thus, it follows that $a_3 = a_4 = 0$ and

$$b_3 = -(n-2) b_4 = \frac{1}{2}(n-2)(n-3) b_5,$$
$$c_3 = -(n-2) c_4 = \frac{1}{2}(n-2)(n-3) c_5. \qquad \text{(A.44)}$$

From these observations, the transverse eigenvalues are given by the secular equations

$$\lambda^{(\mathrm{T})} b_3 = (\mathcal{G} - 2\mathcal{H} + \mathcal{I}) b_3 + (\mathcal{J} - 2\mathcal{K} + \mathcal{L}) c_3, \qquad \text{(A.45)}$$

$$\lambda^{(\mathrm{T})} c_3 = (\mathcal{J} - 2\mathcal{K} + \mathcal{L}) b_3 + (\mathcal{M} - 2\mathcal{N} + \mathcal{O}) c_3, \qquad \text{(A.46)}$$

which are independent of n.

References

1. A. Young, *Spin Glasses and Random Fields* (World Scientific, Singapore, 1997).
2. K. Binder, A. Young, Rev. Mod. Phys. **58**, 801 (1986).
3. K. Fischer, J. Hertz, *Spin Glasses* (Cambridge University Press, Cambridge, 1991).
4. See C. de Dominicis, I. Kondor, T. Temesvari, in reference [1].
5. F.A. da Costa, C.S.O. Yokoi, F.D. Nobre, J. Phys. A **30**, 2317 (1997).
6. F.A. da Costa, C.S.O. Yokoi, S.R.A. Salinas, J. Phys. A **27**, 3365 (1994).
7. J.J. Arenzon, M. Nicodemi, M. Sellitto, J. Phys. I France **6**, 1143 (1996).
8. M. Sellitto, M. Nicodemi, J.J. Arenzon, J. Phys. I France **7**, 945 (1997).
9. G.R. Schreiber, Eur. Phys. J. B **9**, 479 (1999).
10. E.A. Luchinskaya, V.N. Ryzhov, E.E. Tareyeva, J. Phys. C **17**, L665 (1984).
11. T.K. Kopeć, Phys. Rev. B **48**, 3698 (1993).
12. T.K. Kopeć, Phys. Rev. B **48**, 15658 (1993).
13. E.A. Lutchinskaia, E.E. Tareyeva, Phys. Rev. B **52**, 366 (1995).
14. K. Walasek, Phys. Rev. B **51**, 9314 (1995).
15. L.D. Cesare, K. Lukierska-Walasek, I. Rabuffo, K. Walasek, Phys. Rev. B **54**, 15892 (1996).
16. K. Walasek, K. Lukierska-Walasek, M. Wodawski, Acta Phys. Pol. A **92**, 407 (1997).
17. L.D. Cesare, K. Lukierska-Walasek, I. Rabuffo, K. Walasek, Acta Phys. Pol. A **92**, 411 (1997).
18. M. Blume, V.J. Emery, R.B. Griffiths, Phys. Rev. A **4**, 1071 (1971).
19. S. Inawashiro, N.E. Frankel, C.J. Thompson, J. Mag. Mag. Mat. **31-34**, 1297 (1983).
20. S. Inawashiro, N.E. Frankel, C.J. Thompson, Phys. Rev. B **24**, 6532 (1982).
21. S. Inawashiro, N.E. Frankel, C.J. Thompson, Phys. Rev. B **24**, 6542 (1981).
22. S. Inawashiro, N.E. Frankel, C.J. Thompson, Phys. Rev. B **26**, 379 (1982).
23. C.J. Thompson, S. Inawashiro, N.E. Frankel, Prog. Theor. Phys. **87**, 155 (1986).
24. D. Sherrington, S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).
25. J.R.L. de Almeida, D.J. Thouless, J. Phys. A **11**, 983 (1978).
26. S.K. Ghatak, D. Sherrington, J. Phys. C **10**, 3149 (1977).
27. E.J.S. Lage, J.R.L. de Almeida, J. Phys. C **15**, L1187 (1982).
28. P. Mottishaw, D. Sherrington, J. Phys. C **18**, 5201 (1985).
29. See M. Mézard, G. Parisi, M.A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
30. C.D. Dominicis, I. Kondor, in *Lecture Notes in Physics*, edited by L. Garrido (Springer Verlag, Berlim, 1985), Vol. 216.